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# SOME INTEGRAL EQUATIONS RELATED TO ABEL'S EQUATION AND THE HILBERT TRANSFORM

A. S. Peters

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Abstract.

This report is concerned with the inversion of a generalization of the Hilbert transform

$$\int_a^b \frac{\phi(z)dz}{z-\zeta} = f(\zeta)$$

where $f(\zeta)$ is prescribed. The generalized transform is

$$\int_{L_{\alpha\zeta}} \frac{P(\zeta-z)q(z)\phi(z)dz}{(\zeta-z)^{\nu}} + \int_{L_{\zeta\beta}} \frac{P(\zeta-z)r(z)\phi(z)dz}{(z-\zeta)^{\nu}} = f(\zeta)$$

in which $P(z)$ is a polynomial while $q(z)$, $r(z)$ and the path $L_{\alpha\beta}$ in the complex z-plane are arbitrary.

## 1. Introduction

In a paper published in 1922, T. Carleman [1], solved the integral equation

$$\int_a^b \frac{\phi(x)dx}{|\xi-x|^\nu} = f(\xi) \qquad \begin{matrix} a < \xi < b \\ 0 < \nu < 1 \end{matrix} . \tag{1.1}$$

This equation may be regarded as a Fredholm counterpart of Abel's equation

$$\int_a^\xi \frac{\phi(x)dx}{(\xi-x)^\nu} = f(\xi). \tag{1.2}$$

Carleman's analysis implies that

$$F(w) = \int_a^b \frac{\phi(x)dx}{(w-x)^\nu} , \tag{1.3}$$

a function of the complex variable w, can be used to reduce the solution of (1.1) to the solution of a Riemann-Hilbert boundary value problem which can be solved, as Carleman showed, by employing the Plemelj formulas. In 1960 Sakalyuk [2] noted that Carleman's complex variable techniques can be used to furnish a formula for the solution of

$$q(\xi) \int_a^\xi \frac{\phi(x)dx}{(\xi-x)^\nu} + r(\xi) \int_\xi^b \frac{\phi(x)dx}{(x-\xi)^\nu} = f(\xi) . \tag{1.4}$$

Later, Sakalyuk [3], [4] used similar techniques to solve

$$q(\xi) \int_a^\xi \frac{P(\xi-x)\phi(x)dx}{(\xi-x)^\nu} + r(\xi) \int_\xi^b \frac{P(\xi-x)\phi(x)dx}{(x-\xi)^\nu} = f(\xi) , \tag{1.5}$$

where $P(\xi-x)$ is a polynomial; and

$$(1.6) \qquad \int_a^{\xi} \frac{q(x)\phi(x)dx}{(\xi-x)^{\nu}} + \int_{\xi}^{b} \frac{r(x)\phi(x)dx}{(x-\xi)^{\nu}} = f(\xi) \ .$$

More recently, Lundgren and Chiang [5] used transform and Wiener-Hopf procedures to find analytical solutions for the equations

$$(1.7) \qquad \int_a^{\xi} \frac{\phi(x)dx}{(\xi-x)^{\nu}} \pm \int_{\xi}^{b} \frac{\phi(x)dx}{(x-\xi)^{\nu}} = f(\xi)$$

which are particular cases of (1.4).

Equations of the above types arise in various applications. For example, the solution of the equation

$$(1.8) \qquad y^m \chi_{xx}(x,y) + \chi_{yy}(x,y) = 0$$

subject to $y > 0$; and the boundary conditions

$$\chi(x,0) = g(x) \ , \ |x| < 1 \ ;$$

$$\chi_y(x,0) = 0 \quad , \ |x| > 1 \ ,$$

can be reduced to the solution of the integral equation (1.1). The equation (1.4) also occurs in other problems involving Tricomi's equation (1.8); and it occurs in connection with certain hydromagnetic and hydrodynamic flow problems. For expositions see von Wolfensdorf [6], Lundgren and Chiang [5], [7]; and Kirkwood and Rieseman [8].

The equation

$$(1.9) \qquad \int_a^{\xi} \frac{\phi(x)dx}{(\xi-x)^{\nu}} - \int_{\xi}^{b} \frac{\phi(x)dx}{(x-\xi)^{\nu}} = f(\xi)$$

can be regarded as a generalization or modification of the Hilbert transform

$$(1.10) \qquad \int_a^b \frac{\phi(x)dx}{x-\xi} = H_1(\xi) \ .$$

From this point of view, the equation

$$(1.11) \qquad \int_{-\infty}^{\xi} \frac{\phi(x)dx}{(\xi-x)^{\nu}} - \int_{\xi}^{\infty} \frac{\phi(x)dx}{(x-\xi)^{\nu}} = H_1(\xi)$$

recently attracted the interest of Okikiolu [9], [10]; Heywood [11], and Kober [12], [13].

In the above equations the path of integration is a straight line segment along the real axis. The purpose of this paper is to show how a method, different from that used by Sakalyuk, can be used to solve (1.6) and a generalization of it in which the path of integration is situated in an arbitrary position in the complex plane and is a simple smooth curve which may be open or closed.

Section 2 is concerned with the solution of equations related to the generalized Abel equation

$$\int_{L_{\alpha\zeta}} \frac{\phi(z)dz}{(z-\zeta)^{\nu}} = f(\zeta)$$

in which $L_{\alpha\zeta}$ denotes a path in the complex plane. Section 3 demonstrates that the equation

$$(1.12) \qquad \int_{L_{\alpha\zeta}} \frac{q(z)\phi(z)dz}{(\zeta-z)^{\nu}} + \int_{L_{\zeta\beta}} \frac{r(z)\phi(z)dz}{(\zeta-z)^{\nu}} = f(\zeta)$$

is equivalent to an equation with a known solution formula, namely a singular integral equation of the third kind with a Cauchy kernel. Section 4 gives a method for solving

$$(1.13) \qquad \int_{C_{\alpha\zeta}} \frac{\phi(z)dz}{(\zeta-z)^{\nu}} + k\int_{C_{\zeta\alpha}} \frac{\phi(z)dz}{(\zeta-z)^{\nu}} = f(\zeta)$$

where C is a closed curve. The results of Section 4 are used in Section 5 to solve

$$\int_0^1 \frac{\psi(x)\ \mathrm{sgn}\ (\xi-x)dx}{|\sin \pi(\xi-x)|^{\nu}} = g(\xi)$$

and in Section 6 to find the inversion formula for the generalized Hilbert transform

$$\int_{-\infty}^{\infty} \frac{\phi(x)\ \mathrm{sgn}\ (\xi-x)dx}{|\xi-x|^{\nu}} = f(\xi)\ .$$

In Section 7 we indicate how to solve a generalization of (1.12) namely

$$\int_{L_{\alpha\zeta}} \frac{P(\zeta-z)q(z)\phi(z)dz}{(\zeta-z)^{\nu}} + \int_{L_{\zeta\beta}} \frac{P(\zeta-z)r(z)\phi(z)dz}{(\zeta-z)^{\nu}} = f(\zeta)$$

where P(t) is a polynomial.

## 2. A Generalization of Abel's Equation

The methods given in the following sections involve the solutions of the equation

$$(2.1) \qquad \int_{L_{\alpha\zeta}} (\zeta - z)^{m-\nu}\, \phi(z)\,dz = f(\zeta)$$

where m is zero or a positive integer and $0 < \mathcal{R}e\ \nu < 1$; that is, the real part of $\nu$ is between zero and one. We use the notation $L_{\alpha\beta}$ to denote a fixed, directed, simple smooth curve, or path, which begins at $\alpha$, ends at $\beta$, and lies in the complex z-plane. Thus $L_{\alpha\zeta}$ denotes the path along $L_{\alpha\beta}$ from $\alpha$ to $\zeta$, an arbitrary point on $L_{\alpha\beta}$. The prescribed function $f(\zeta)$ is supposed to be integrable for $\zeta$ on $L_{\alpha\beta}$ and to possess whatever properties are implied by the integral on the left hand side of (2.1) when the unknown complex function $\phi(z)$ is integrable along $L_{\alpha\beta}$.

The equation (2.1) can be solved by multiplying it by $(z-\zeta)^{\nu-1}$ and integrating along $L_{\alpha z}$. This gives

$$\int_{L_{\alpha z}} \frac{1}{(z-\zeta)^{1-\nu}} \int_{L_{\alpha\zeta}} \frac{(\zeta-w)^{m}\phi(w)\,dw}{(\zeta-w)^{\nu}}\,d\zeta = \int_{L_{\alpha z}} \frac{f(\zeta)\,d\zeta}{(z-\zeta)^{1-\nu}}$$

or

$$\int_{L_{\alpha\beta}} \frac{\mu(z-\zeta)}{(z-\zeta)^{1-\nu}} \int_{L_{\alpha\beta}} \frac{\mu(\zeta-w)\cdot(\zeta-w)^{m}\phi(w)\,dw\,d\zeta}{(\zeta-w)^{\nu}} = \int_{L_{\alpha z}} \frac{f(\zeta)\,d\zeta}{(z-\zeta)^{1-\nu}}$$

where $(z-\zeta)$ is defined by

$$\mu(z-\zeta) = \begin{cases} 1 & \text{if } z \text{ follows } \zeta \text{ along } L_{\alpha\beta} \\ 0 & \text{if } z \text{ precedes } \zeta \text{ along } L_{\alpha\beta}. \end{cases}$$

From an interchange of order of integration we have

$$\int_{L_{\alpha z}} \phi(w) \int_{L_{wz}} \frac{(\zeta-w)^m d\zeta\, dw}{(z-\zeta)^{1-\nu}(\zeta-w)^{\nu}} = \int_{L_{\alpha z}} \frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\nu}}$$

which leads to

$$\frac{\Gamma(m-\nu+1)\Gamma(\nu)}{\Gamma(m+1)} \int_{L_{\alpha z}} (z-w)\phi(w)dw = \int_{L_{\alpha z}} \frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\nu}} .$$

If we differentiate the last result we find

$$(2.2) \qquad \phi(z) = \frac{1}{\Gamma(m-\nu+1)\Gamma(\nu)} \cdot \frac{d^{m+1}}{dz^{m+1}} \int_{L_{\alpha z}} \frac{f(\zeta)d\xi}{(z-\zeta)^{1-\nu}} ,$$

and we see that the solution of (2.1) has the same form that it has when $L_{\alpha\beta}$ is a line segment along the real axis.

In a similar way, or by introducing translations which reduce (2.3)-(2.4) to (2.1)-(2.2), it can be shown that we have the pair

$$(2.3) \qquad \int_{L_{\zeta\beta}} (z-\zeta)^{m-\nu}\, \phi(z)\, dz = f(\zeta)$$

$$(2.4) \qquad \phi(z) = \frac{(-1)^{m+1}}{\Gamma(m-\nu+1)\Gamma(\nu)} \frac{d^{m+1}}{dz^{m+1}} \int_{L_{z\beta}} \frac{f(\zeta)d\zeta}{(\zeta-z)^{1-\nu}} .$$

It is also easy to see that the last pair implies the pair

$$(2.5) \qquad \int_{L_{\zeta\beta}} (\zeta-z)^{m-\nu}\phi(z)\, dz = f(\zeta)$$

$$\phi(z) = \frac{1}{\Gamma(m-\nu+1)\Gamma(\nu)} \frac{d^{m+1}}{dz^{m+1}} \int_{L_{z\beta}} \frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\nu}} \quad . \tag{2.6}$$

It should be noted that, with appropriate restrictions on $f(\zeta)$ and $\phi(z)$ with respect to integrability along a path which extends to infinity, the above results are valid if either of the end points of $L_{\alpha\beta}$ is at infinity. For example, if each of $f(\zeta)$ and $\phi(z)$ is integrable along $L_{\alpha\,\infty}$ we find that the solution of

$$\int_{L_{\zeta\,\infty}} \frac{\phi(z)dz}{(z-\zeta)^{\nu}} = f(\zeta) \tag{2.7}$$

is given by

$$\begin{aligned} \phi(z) &= -\frac{1}{\Gamma(1-\nu)\Gamma(\nu)} \frac{d}{dz} \int_{L_{z\,\infty}} \frac{f(\zeta)d\zeta}{(\zeta-z)^{1-\nu}} \\ &= -\frac{\sin \pi\nu}{\pi} \frac{d}{dz} \int_{L_{z\,\infty}} \frac{f(\zeta)d\zeta}{(\zeta-z)^{1-\nu}} \quad . \end{aligned} \tag{2.8}$$

## 3. An Integral Equation of the First Kind with the Kernel $[q(z)\,\mu(\zeta-z) + r(z)\,\mu(z-\zeta)]/(\zeta-z)^{\nu}$.

We proceed to show that the solution of the integral equation

$$\int_{L_{\alpha\zeta}} \frac{q(z)\phi(z)dz}{(\zeta-z)^{\nu}} + \int_{L_{\zeta\beta}} \frac{r(z)\phi(z)dz}{(\zeta-z)^{\nu}} = f(\zeta) \tag{3.1}$$

must also satisfy a singlular integral equation of the third kind with a kernel of the Cauchy type. We suppose that each of $q(z)$ and $r(z)$ is a single valued complex function which satisfies a uniform Holder condition on $L_{\alpha\beta}$. We interpret $(\zeta-z)^{\nu}$, $(0 < \mathcal{R}e\ \nu < 1)$, to mean

$$(\zeta-z)^{\nu} = \exp\ \{\nu\ \ell n\ |\zeta-z| + i\ \nu\ \arg\ (\zeta-z)\}$$

where $\arg\ (\zeta-z)$ is taken to be continuous for $\zeta$, $z$ on $L_{\alpha\beta}$. We also assume that the unknown complex function $\phi(z)$ is integrable on $L_{\alpha\beta}$ while it satisfies a Holder continuity condition (not necessarily uniform) on $L'_{\alpha\beta}$ which means $L_{\alpha\beta}$ minus its end points:- symbolically,

$$L'_{\alpha\beta} = L_{\alpha\beta} - \alpha - \beta.$$

Singularities of $\phi(z)$ at the end points are not precluded. For example, in the neighborhood of $z = \alpha$, $\phi(z)$ may behave like $(z-\alpha)^{\xi-1}$, $0 < \mathcal{R}e\ \xi < 1$; or like $\ell n\ (z-\alpha)$. In addition we assume that the prescribed function $f(\zeta)$ possesses properties which match those of the sum of the integrals in (3.1); and these properties admit possible singularities for $f(\zeta)$ at the end points of $L_{\alpha\beta}$. The above assumptions can be relaxed but for simplicity we will retain them in what follows. The Holder conditions are imposed in order to guarantee the existence of integrals of the type

$$\int_{L_{\alpha\beta}} \frac{r(z)\phi(z)dz}{z-\zeta} \quad \zeta \text{ on } L'_{\alpha\beta}$$

where here and in what follows the symbol

$$\int_{L_{\alpha\beta}} \frac{\psi(z)dz}{z-\zeta} \qquad \zeta \text{ on } L'_{\alpha\beta}$$

is to be interpreted as the Cauchy principal value.

If we were contemplating an analysis of (3.1) in strict accordance with Carleman's ideas we would seek a function $F(w)$ such that (3.1) requires this function to satisfy a Riemann-Hilbert boundary value problem whose solution determines $\phi(z)$. For (3.1), however, it is not clear, at least a priori, what we should take $F(w)$ to be and the choice seems even more obscure when $L_{\alpha\beta}$ is closed. For this reason we are going to use another method which also differs from the one Sakalyuk [4] used for the case in which $L_{\alpha\beta}$ is a segment along the real axis.

We are going to base the analysis of (3.1) on the following integral:-

$$(3.2) \quad \int_{L_{\alpha\zeta}} \frac{d\lambda}{(\lambda-\alpha)^{1-\nu}(\zeta-\lambda)^{\nu}(\lambda-z)}$$

$$= \begin{cases} \dfrac{-\pi \cot \pi\nu}{(z-\alpha)^{1-\nu}(\zeta-z)^{\nu}} & z \text{ on } L'_{\alpha\zeta} = L_{\alpha\beta} - L_{\zeta\beta} \\ \dfrac{-\pi e^{-\pi i\nu}}{(z-\alpha)^{1-\nu}(\zeta-z)^{\nu} \sin \pi\nu} & z \text{ not on } L_{\alpha\zeta}. \end{cases}$$

This evaluation shows that

$$\psi(\lambda) = \frac{1}{(\lambda-\alpha)^{1-\nu}(\zeta-\lambda)^{\nu}}$$

satisfies the equation

$$(3.3) \qquad \int_{L_{\alpha\zeta}} \frac{\psi(\lambda)d\lambda}{\lambda - z} = \gamma\, \psi(z) \quad z \text{ on } L'_{\alpha\zeta}$$

where $\gamma = -\pi \cot \pi\nu$. We can say that, in a sense, we are going to base the analysis of (3.1) on (3.3). In the sequel we will show that if each of $q(z)$ and $r(z)$ is constant then a knowledge of (3.2) and the solution of Abel's equation is sufficient for finding the solution of (3.1).

The result of multiplying (3.2) by $(z-\alpha)^{1-\nu} r(z)\, \phi(z)$ and then integrating with respect to z along $L_{\alpha\beta}$ is the equation

$$\int_{L_{\zeta\beta}} \frac{r(z)\phi(z)dz}{(\zeta - z)^{\nu}} = -\, e^{\pi i\nu} \cdot \cos \pi\nu \cdot \int_{L_{\alpha\zeta}} \frac{r(z)\phi(z)dz}{(\zeta - z)^{\nu}}$$

$$(3.4) \qquad + \frac{e^{\pi i\nu} \sin \pi\nu}{\pi} \int_{L_{\alpha\zeta}} \frac{1}{(\zeta - \lambda)^{\nu}(\lambda - \alpha)^{1-\nu}} \int_{L_{\alpha\beta}} \frac{(z-\alpha)^{1-\nu} r(z)\phi(z)dzd\lambda}{z - \lambda} .$$

The elimination of

$$\int_{L_{\zeta\beta}} \frac{r(z)\phi(z)dz}{(\zeta - z)^{\nu}}$$

from (3.1) and (3.4) gives

$$(3.5) \int_{L_{\alpha\zeta}} \frac{1}{(\zeta - z)^{\nu}} \left\{ \begin{array}{l} [q(z) - e^{\pi i\nu} r(z) \cos \pi\nu]\phi(z) \\ \\ + \dfrac{e^{\pi i\nu} \sin \pi\nu}{\pi(z-\alpha)^{1-\nu}} \displaystyle\int_{L_{\alpha\beta}} \frac{(\lambda-\alpha)^{1-\nu} r(\lambda)\phi(\lambda)d\lambda}{\lambda - z} \end{array} \right. dz = f(\zeta).$$

This is a generalized Abel equation, and by using the formula (2.2) with m = 0 we see that if $\phi(z)$ satisfies (3.1) then it must also satisfy

$$[q(z) - e^{\pi i \nu} r(z) \cos \pi \nu]\phi(z)$$

$$= \frac{\sin \pi\nu}{\pi} \frac{d}{dz} \int_{L_{\alpha z}} \frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\nu}} \tag{3.6}$$

$$+ \frac{e^{\pi i \nu} \sin \pi\nu}{\pi(z-\alpha)^{1-\nu}} \int_{L_{\alpha\beta}} \frac{(\lambda-\alpha)^{1-\nu} r(\lambda)\phi(\lambda)d\lambda}{\lambda - z}$$

Conversely, if $\phi(z) = \phi_1(z)$ is any solution of (3.6) which satisfies the conditions already imposed on $\phi$ and is such that, with possible singularities at $z = \alpha$ and $z = \beta$, the integrals

$$\int_{L_{\alpha\zeta}} \frac{q(z)\phi_1(z)dz}{(\zeta-z)^{\nu}} \text{ and } \int_{L_{\zeta\beta}} \frac{r(z)\phi_1(z)dz}{(\zeta-z)^{\nu}}$$

exist then we can reverse the above manipulations and pass from (3.6) to (3.1). Therefore, the equation (3.6), subject to the conditions mentioned above, is equivalent to (3.1).

The solution of (3.6) is well known. The equation (3.6) can be solved by using Carleman's function theoretic method. Expositions of this method can be found in the text by Muskhelishivili [14] and in the text by Gakhov [15]. It is sufficient for our purposes to record here only the formula for the solution of the equation of the second kind, namely

$$(3.7) \qquad \int_{L_{\alpha\beta}} \frac{\psi(\lambda)d\lambda}{\lambda - z} = \tau\,\psi(z) + g(z).$$

The solution of (3.7) is given by

$$(3.8) \quad \psi(z) = \frac{-\sin^2 \gamma\pi}{\pi^2 (z-\alpha)^{1-\gamma}} \frac{d}{dz} \int_{L_{z\beta}} \frac{1}{(w-z)^{\gamma}} \int_{L_{\alpha w}} \frac{(\zeta-\alpha)^{1-\gamma} g(\zeta)d\zeta dw}{(w-\zeta)^{1-\gamma}}$$

$$+ \frac{c \sin \gamma\pi}{\pi (z-\alpha)^{1-\gamma}(\beta-z)^{\gamma}}$$

or, in another form, by

$$(3.9) \quad \psi(z) = \frac{-\tau g(z)}{\tau^2+\pi^2} - \frac{1}{(\tau^2+\pi^2)(z-\alpha)^{1-\gamma}(\beta-z)^{\gamma}} \int_{L_{\alpha\beta}} \frac{(\zeta-\alpha)^{1-\gamma}(\beta-\zeta)^{\gamma} g(\zeta)d\zeta}{\zeta - z}$$

$$+ \frac{c}{(z-\alpha)^{1-\gamma}(\beta-z)^{\gamma}\sqrt{\tau^2+\pi^2}}$$

where $\gamma$ must satisfy

$$-\pi \cot \gamma\pi = \tau \quad , \quad 0 < \mathrm{Re}\ \gamma < 1$$

and

$$c = \int_{L_{\alpha\beta}} \psi(z)dz.$$

We see from (3.8) or (3.9) that the solution of (3.7) is not unique. However, the arbitrary constant c can be fixed by imposing a side condition on $\psi(z)$. The solution of (3.7) which is bounded at $z = \alpha$ is

$$(3.10)\quad \psi(z) = \frac{-\tau\, g(z)}{\tau^2+\pi^2} - \frac{(z-\alpha)^\gamma}{(\tau^2+\pi^2)(\beta-z)^\gamma} \int_{L_{\alpha\beta}} \frac{(\beta-\zeta)^\gamma g(\zeta)d\zeta}{(\zeta-\alpha)^\gamma(\zeta-z)} .$$

It may be interesting to note that (3.8) can be found by a process which relies only on the integral (3.2) and the solution of Abel's equation:- See Peters [16] and Williams [17].

As an illustration let us consider the solution of

$$(3.11)\qquad \int_{L_{\alpha\zeta}} \frac{\phi(z)dz}{(\zeta-z)^\nu} - \int_{L_{\zeta\beta}} \frac{\phi(z)dz}{(z-\zeta)^\nu} = f(\zeta).$$

For this equation we suppose that $(\zeta-z)^\nu$ in the first integral varies continuously with z on $L_{\alpha\zeta}$ and that

$$\lim_{z\longrightarrow\zeta} \arg\,(\zeta-z) = \theta(\zeta)$$

where $\theta(\zeta)$ is the continuously varying argument or angle of the unit tangent vector to $L_{\alpha\beta}$ at $z = \zeta$ subject to $0 \leq \theta(\alpha) < 2\pi$. Likewise, $(z-\zeta)^\nu$ in the second integral of (3.11) is supposed to vary continuously with z on $L_{\zeta\beta}$ and

$$\lim_{z\,-\,\zeta} \arg\,(z-\zeta) = \theta(\zeta).$$

The equation (3.11) is a particular case of (3.1). It is generated by (3.1) if we take

$$(\zeta-z)^\nu = e^{-\pi i\nu}(z-\zeta)^\nu$$

for z on $L_{\zeta\beta}$, $r(z) = -e^{\pi i\nu}$ and $q(z) = 1$. Then from (3.6) the

solution of (3.11) must satisfy

$$\frac{\pi(1+\cos \pi\nu)}{\sin \pi\nu}\, \phi(z)\cdot(z-\alpha)^{1-\nu}$$

$$(3.12)\quad \int_{L_{\alpha\beta}} \frac{(\lambda-\alpha)^{1-\nu}\phi(\lambda)d\lambda}{\lambda-z} =$$

$$-\,(z-\alpha)^{1-\nu}\,\frac{d}{dz}\int_{L_{\alpha z}} \frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\nu}}.$$

In accordance with (3.9) the solution of (3.12) is

$$(3.13)\qquad \phi(z) = \frac{\sin \pi\nu}{2\pi}\,\frac{d}{dz}\int_{L_{\alpha z}} \frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\nu}}$$

$$+\,\frac{\sin^2 \frac{\pi\nu}{2}}{\pi^2(z-\alpha)^{1-\frac{\nu}{2}}(\beta-z)^{1-\frac{\nu}{2}}}\int_{L_{\alpha\beta}} \frac{(\zeta-\alpha)^{1-\frac{\nu}{2}}(\beta-\zeta)^{1-\frac{\nu}{2}}}{\zeta-z}\,\frac{d}{d\zeta}\int_{L_{\alpha\zeta}} \frac{f(\tau)d\tau d\zeta}{(\zeta-\tau)^{1-\nu}}$$

$$+\,\frac{c\,\sin \frac{\pi\nu}{2}}{\pi(z-\alpha)^{1-\frac{\nu}{2}}(\beta-z)^{1-\frac{\nu}{2}}}$$

where

$$c = \int_{L_{\alpha\beta}} \phi(z)(z-\alpha)^{1-\nu}\, dz \quad .$$

However, before we can accept this as the solution of (3.11) we need to examine the nature of the singularity at $z = \alpha$. If $f \equiv 0$, (3.13) becomes

$$\phi(z) = \frac{c \sin \frac{\pi\nu}{2}}{\pi(z-\alpha)^{1-\frac{\nu}{2}}(\beta-z)^{1-\frac{\nu}{2}}} \tag{3.14}$$

$$0 < \mathrm{Re}\ \nu < 1.$$

The singularities of (3.14) are admissible because

$$\frac{1}{(z-\alpha)^{1-\frac{\nu}{2}}(\beta-z)^{1-\frac{\nu}{2}}(\zeta-z)^{\nu}}$$

is integrable with respect to z along $L_{\alpha\beta}$; and in fact substitution in (3.11) shows that (3.14) satisfies (3.11). Furthermore with $\phi(z)$ integrable along $L_{\alpha\beta}$ the singularity of $f(\zeta)$ at $\zeta = \alpha$ can be no worse than that given by

$$\frac{\text{const.}}{(\zeta-\alpha)^{\nu-\varepsilon}}$$

where $0 < \mathrm{Re}\ \varepsilon < 1$. Therefore it can be verified that the singularity at $z = \alpha$ of the sum of the first two terms on the right hand side of the equality sign in (3.13) can be no worse than

$$\frac{\text{const.}}{(z-\alpha)^{1-\varepsilon}} .$$

Hence it turns out that (3.13) is actually the general solution of (3.11).

The appearance of the constant c in (3.13) shows that the solution of (3.11) is not unique but the prescription of some side conditions on $\phi$ leads to a unique solution. Another form for the solution of (3.11) can be exhibited if we use (3.8)

instead of (3.9). Still another form can be found in a paper by Lundgren and Chiang [5] where the path $L_{\alpha\beta}$ is assumed to be a line segment along the real axis.

In contrast to the solution of (3.11), the solution of

$$(3.15) \qquad \int_{L_{\alpha\zeta}} \frac{\phi(z)dz}{(\zeta-z)^{\nu}} + \int_{L_{\zeta\beta}} \frac{\phi(z)dz}{(z-\zeta)^{\nu}} = f(\zeta)$$

is unique. Here, the corresponding singular integral equation of Cauchy type is

$$(3.16) \qquad \int_{L_{\alpha\beta}} \frac{(\lambda-\alpha)^{1-\nu}\phi(\lambda)d\lambda}{\lambda-z} = \frac{\pi[\cos \pi\nu - 1]}{\sin \pi\nu}\,\phi(z)(z-\alpha)^{1-\nu} + (z-\alpha)^{1-\nu}\,\frac{d}{dz}\int_{L_{\alpha z}} \frac{f(\zeta)}{(z-\zeta)^{1-\nu}}\frac{d\zeta}{1-\nu}.$$

For this case the quantity $\gamma$ which appears in the solution formula (3.9) is fixed by

$$-\pi \cot \gamma\pi = \frac{\pi[\cos \pi\nu - 1]}{\sin \pi\nu}$$

$$0 < \mathrm{Re}\ \gamma < 1$$

which gives

$$\gamma = \frac{1-\nu}{2} .$$

Therefore the arbitrary term in the solution of (3.16) is

$$(z-\alpha)^{1-\nu}\phi(z) = \frac{\text{const.}}{(z-\alpha)^{\frac{1+\nu}{2}}(\beta-z)^{\frac{1-\nu}{2}}}$$

which leads to

$$\phi(z) = \frac{\text{const.}}{(z-\alpha)^{\frac{3-\nu}{2}}(\beta-z)^{\frac{1-\nu}{2}}}.$$

This function, however, cannot be admitted as a solution of (3.15) because its behaviour in the neighborhood of $z = \alpha$ prohibits integration. We are compelled therefore to use the formula (3.10). This leads to

$$\phi(z) = \frac{\sin \pi\nu}{2\pi} \frac{d}{dz} \int_{L_{\alpha z}} \frac{f(\zeta)\, d\zeta}{(z-\zeta)^{1-\nu}}$$

(3.17)

$$- \frac{\cos^2 \frac{\pi\nu}{2}}{\pi^2} \cdot \frac{(z-\alpha)^{\frac{\nu-1}{2}}}{(\beta-z)^{\frac{1-\nu}{2}}} \int_{L_{\alpha\beta}} \frac{(\beta-\zeta)^{\frac{1-\nu}{2}}}{(\zeta-\alpha)^{\frac{\nu-1}{2}}(\zeta-z)} \frac{d}{d\zeta} \int_{L_{\alpha\zeta}} \frac{f(\tau)\, d\tau\, d\zeta}{(\zeta-\tau)^{1-\nu}}$$

for the solution of (3.15).

The solution (3.17) can be presented in other forms which may be useful for certain particular calculations. Alternate expressions for the case in which $L_{\alpha\beta}$ is a line segment along the real axis can be found in papers by Carleman [1], Williams [17]; Lundgren and Chiang [5].

## 4. A Case for which $L_{\alpha\beta}$ is a Simple Closed Path.

If $L_{\alpha\beta}$ is a simple, smooth closed path C directed counter clockwise; and if $\alpha$, an arbitrary point on C, is used to serve as both initial and terminal point of C the equation (3.1) becomes

$$(4.1)\qquad \int_{C_{\alpha\zeta}} \frac{q(z)\phi(z)dz}{(\zeta-z)^{\nu}} + \int_{C_{\zeta\alpha}} \frac{r(z)\phi(z)dz}{(\zeta-z)^{\nu}} = f(\zeta).$$

This equation, subject to the conditions imposed in Section 3, is, as we have shown there, equivalent to

$$(4.2)\qquad \begin{aligned} &[q(z) - e^{\pi i\nu} r(z) \cos \pi\nu]\phi(z) \\ &+ \frac{e^{\pi i\nu} \sin \pi\nu}{\pi(z-\alpha)^{1-\nu}} \oint_C \frac{(\lambda-\alpha)^{1-\nu} r(\lambda)\phi(\lambda)d\lambda}{\lambda-z} \\ &= \frac{\sin \pi\nu}{\pi} \frac{d}{dz} \int_{C_{\alpha z}} \frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\nu}} . \end{aligned}$$

In general, it is easier to solve (4.2) than it is to solve (3.6). This is especially true if $q(z)$ and $r(z)$ satisfy certain analyticity conditions on C and its interior:- (see Peters [18]).

If we set $q(z) \equiv 1$ and $r(z) \equiv k$, a constant, in (4.1) and (4.2) we have the equations

$$(4.3)\qquad \int_{C_{\alpha\zeta}} \frac{\phi(z)dz}{(\zeta-z)^{\nu}} + k \int_{C_{\zeta\alpha}} \frac{\phi(z)dz}{(\zeta-z)^{\nu}} = f(\zeta)$$

and

$$(4.4)\qquad \begin{aligned} &[1-ke^{\pi i\nu} \cos \pi\nu]\phi(z) + \frac{ke^{\pi i\nu} \sin \pi\nu}{\pi(z-\alpha)^{1-\nu}} \oint_C \frac{(\lambda-\alpha)^{1-\nu}}{\lambda-z}\phi(\lambda)d\lambda \\ &= \frac{\sin \pi\nu}{\pi} \frac{d}{dz} \int_{C_{\alpha z}} \frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\nu}} . \end{aligned}$$

We are now going to show that the solution of (4.3) can be found without actually solving a Cauchy integral equation.

It turns out that the solution of (4.3) can be based entirely on the integral (3.2) and another like it. If we use

$$(4.5)\quad \int_{C_{\zeta\alpha}} \frac{d\lambda}{(\lambda-\alpha)^{1-\nu}(\zeta-\lambda)^{\nu}(\lambda-z)} = \begin{cases} \dfrac{\pi e^{\pi i\nu}}{(z-\alpha)^{1-\nu}(\zeta-z)^{\nu}\sin \pi\nu} & z \text{ on } C'_{\alpha\zeta} \\ \dfrac{\pi \cot \pi\nu}{(z-\alpha)^{1-\nu}(\zeta-z)^{\nu}} & z \text{ on } C'_{\zeta\alpha} \end{cases}$$

we can reduce (4.3) to an equation of the same form as (4.4) and then the elimination of the unknown integral from the equations provides the solution. After multiplying (4.5) by $(z-\alpha)^{1-\nu}\phi(z)$ and integrating along C we find

$$(4.6)\quad \int_{C_{\alpha\zeta}} \frac{\phi(z)dz}{(\zeta-z)^{\nu}} = -\frac{e^{-\pi i\nu}\sin \pi\nu}{\pi} \int_{C_{\zeta\alpha}} \frac{1}{(\zeta-\lambda)^{\nu}(\lambda-\alpha)^{1-\nu}} \oint_{C} \frac{(z-\alpha)^{1-\nu}\phi(z)dzd\lambda}{z-\lambda}$$

$$- e^{-\pi i\nu}\cos \pi\nu \int_{C_{\zeta\alpha}} \frac{\phi(z)dz}{(\zeta-z)^{\nu}} .$$

The elimination of

$$\int_{C_{\alpha\zeta}} \frac{\phi(z)dz}{(\zeta-z)^{\nu}}$$

from (4.6) and (4.3) gives

$$(4.7)\quad \int_{C_{\zeta\alpha}} \frac{1}{(\zeta-z)^{\nu}} \left\{ \begin{array}{l} [k - e^{-\pi i\nu} \cos \pi\nu]\phi(z) \\ \\ - \dfrac{e^{-\pi i\nu} \sin \pi\nu}{\pi(z-\alpha)^{1-\nu}} \oint_C \dfrac{(\lambda-\alpha)^{1-\nu}\phi(\lambda)d\lambda}{\lambda-z} \end{array} \right\} dz = f(\zeta).$$

If we refer to the pair (2.5)-(2.6) we can see that the solution of (4.7) is

$$(4.8)\quad [k - e^{-\pi i\nu} \cos \pi\nu]\phi(z) - \frac{e^{-\pi i\nu} \sin \pi\nu}{\pi(z-\alpha)^{1-\nu}} \oint_C \frac{(\lambda-\alpha)^{1-\nu}\phi(\lambda)d\lambda}{\lambda-z}$$

$$= \frac{\sin \pi\nu}{\pi} \frac{d}{dz} \int_{C_{z\alpha}} \frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\nu}} .$$

From (4.4) and (4.8) we now have the equations

$$(4.9)\quad [e^{-\pi i\nu} - k \cos \pi\nu]\phi(z) + \frac{k \sin \pi\nu}{\pi(z-\alpha)^{1-\nu}} \oint_C \frac{(\lambda-\alpha)^{1-\nu}\phi(\lambda)d\lambda}{\lambda-z}$$

$$= \frac{e^{-\pi i\nu} \sin \pi\nu}{\pi} \frac{d}{dz} \int_{C_{\alpha z}} \frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\nu}}$$

and

$$(4.10)\quad [k^2 e^{\pi i\nu} - k \cos \pi\nu]\phi(z) - \frac{k \sin \pi\nu}{\pi(z-\alpha)^{1-\nu}} \oint_C \frac{(\lambda-\alpha)^{1-\nu}\phi(\lambda)d\lambda}{\lambda-z}$$

$$= \frac{ke^{\pi i\nu} \sin \pi\nu}{\pi} \frac{d}{dz} \int_{C_{z\alpha}} \frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\nu}}$$

which may be regarded as equations for the determination of $\phi(z)$

and $\dfrac{1}{(z-\alpha)^{1-\nu}} \oint_C \dfrac{(\lambda-\alpha)^{1-\nu}\phi(\lambda)d\lambda}{\lambda-z}$. The determinant of this system is

$$D = -k(k-1)(k-e^{-2\pi i\nu})e^{\pi i\nu} \sin \pi\nu.$$

Hence if k and $\nu$ are values for which $D \neq 0$ we find that (4.9) and (4.10) yield

$$[k-1][ke^{\pi i\nu}-e^{-\pi i\nu}]\phi(z)$$

(4.11)

$$= \frac{\sin \pi\nu}{\pi} \frac{d}{dz} \left\{ e^{-\pi i\nu} \int_{C_{\alpha z}} \frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\nu}} + ke^{\pi i\nu} \int_{C_{z\alpha}} \frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\nu}} \right\}$$

which gives the solution of (4.3).

We can now show the solution of

$$\int_{C_{\alpha\zeta}} \frac{\phi(z)dz}{(\zeta-z)^{\nu}} - \int_{C_{\zeta\alpha}} \frac{\phi(z)dz}{(z-\zeta)^{\nu}} = f(\zeta) \tag{4.12}$$

which is a generalization of the basic equation

$$\oint_C \frac{\phi(z)dz}{z-\zeta} = -f(\zeta). \tag{4.13}$$

If in equation (4.3) we take

$$(\zeta-z)^{\nu} = e^{-\pi i\nu}(z-\zeta)^{\nu}$$

when z is on $C_{\zeta\alpha}$, and take $k = -e^{-\pi i\nu}$, then (4.3) reduces to (4.12); and (4.11) shows that

$$(4.14) \qquad \phi(z,\nu) = \frac{1}{2\pi} \tan\frac{\pi\nu}{2} \cdot \frac{d}{dz} \left\{ \begin{array}{l} \displaystyle\int_{C_{\alpha z}} \frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\nu}} \\ \\ \displaystyle + \int_{C_{z\alpha}} \frac{f(\zeta)d\zeta}{(\zeta-z)^{1-\nu}} \end{array} \right\}$$

is the solution of (4.12). As a check it can be verified that if we integrate (4.14) and then take the limit of the resulting equation as $\nu$ is allowed to approach 1, the limit process gives

$$\lim_{\nu \longrightarrow 1} \phi(z,\nu) = \frac{1}{\pi^2} \oint_C \frac{f(\zeta)d\zeta}{\zeta - z}$$

which is the well known solution of

$$\int_{C_\zeta} \frac{\phi(z)dz}{\zeta - z} - \int_{C_\zeta} \frac{\phi(z)dz}{z-\zeta} = f(\zeta)$$

or

$$\oint_C \frac{\phi(z)dz}{z-\zeta} = -f(\zeta).$$

5. An Integral Equation with the Kernel $\operatorname{sgn}(\zeta-x)/|\sin \pi(\zeta-x)|^{\nu}$.

There are some interesting integral equations of the form

$$\int_a^b k(\xi,x)\, \psi(x)\, dx = g(\xi)$$

which can be solved by transforming them into (4.3). One of these is the equation

$$(5.1) \qquad \int_0^1 \frac{\psi(x)\ \mathrm{sgn}\ (\xi-x)dx}{|\sin \pi(\xi-x)|^{\nu}} = g(\xi)$$

$$0 < \nu \leq 1 \quad ; \quad 0 < \xi < 1.$$

If we use the transformations

$$e^{2\pi i x} = z \quad , \quad e^{2\pi i \xi} = \zeta \ ;$$

equation (5.1) becomes

$$(5.2) \qquad \int_{C_{\alpha\zeta}} \frac{\phi(z)dz}{(\zeta-z)^{\nu}} - \int_{C_{\zeta\alpha}} \frac{\phi(z)dz}{(z-\zeta)^{\nu}} = f(\zeta)$$

where C is the unit circle in the z-plane, $\alpha = 1$, and

$$\phi(z) = \frac{(2i)^{\nu} z^{\nu/2}}{2\pi i z} \psi\left(\frac{\ell n\ z}{2\pi i}\right)$$

with

$$f(\zeta) = \frac{1}{\zeta^{\nu/2}}\ g\ \left(\frac{\ell n\ \zeta}{2\pi i}\right) .$$

From (4.14) the solution of (5.2) is

$$(5.3) \qquad \phi(z) = \frac{1}{2\pi} \tan \frac{\pi\nu}{2} \cdot \frac{d}{dz} \left\{ \begin{array}{l} \displaystyle\int_{C_{\alpha z}} \frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\nu}} \\ \\ \displaystyle + \int_{C_{z\alpha}} \frac{f(\zeta)d\zeta}{(\zeta-z)^{1-\nu}} \end{array} \right\} .$$

Transforming this back to the original variables we find that

$$(5.4)\qquad \psi(x) = \frac{1}{2}\tan\frac{\pi\nu}{2}\cdot\left\{\begin{array}{l} \pi i(\nu-1)e^{-\pi i x}\displaystyle\int_0^1 \frac{e^{\pi i\xi}g(\xi)d\xi}{|\sin\pi(x-\xi)|^{1-\nu}} \\ \\ +\, e^{-\pi i x}\dfrac{d}{dx}\displaystyle\int_0^1 \frac{e^{\pi i\xi}g(\xi)d\xi}{|\sin\pi(x-\xi)|^{1-\nu}} \end{array}\right\}$$

and from further manipulation that

$$(5.5)\qquad \psi(x) = \frac{1}{2}\tan\frac{\pi\nu}{2}\cdot\left\{\begin{array}{l} \dfrac{d}{dx}\displaystyle\int_0^1 \frac{\cos\pi(\xi-x)\cdot g(\xi)d\xi}{|\sin\pi(\xi-x)|^{1-\nu}} \\ \\ -\pi\nu\displaystyle\int_0^1 \frac{\sin\pi(\xi-x)\cdot g(\xi)d\xi}{|\sin\pi(\xi-x)|^{1-\nu}} \end{array}\right\}$$

is the solution of (5.1), namely

$$\int_0^1 \frac{\psi(x)\ \operatorname{sgn}(\xi-x)dx}{|\sin\pi(\xi-x)|^{\nu}} = g(\xi),$$

which is a generalization of

$$\int_0^1 \frac{\psi(x)dx}{\sin\pi(\xi-x)} = g(\xi).$$

## 6. Generalizations of the Hilbert Transform.

The function

$$(6.1)\qquad H_1(\xi) = \int_a^b \frac{\psi(x)dx}{x-\xi}$$

or a constant multiple of it is called the Hilbert transform of $\psi(x)$. The function

$$H(\zeta) = \int_{L_{\alpha\beta}} \frac{\psi(z)dz}{\zeta - z} \tag{6.2}$$

$$= \int_{L_{\alpha\zeta}} \frac{\psi(z)dz}{\zeta - z} - \int_{L_{\zeta\beta}} \frac{\psi(z)dz}{z - \zeta}$$

is a generalization of (6.1) and

$$h(\zeta) = \int_{L_{\alpha\zeta}} \frac{\psi(z)dz}{(\zeta - z)^{\nu}} - \int_{L_{\zeta\beta}} \frac{\psi(z)dz}{(z - \zeta)^{\nu}} \tag{6.3}$$

or

$$h(\zeta) = \int_{C_{\alpha\zeta}} \frac{\psi(z)dz}{(\zeta - z)^{\nu}} - \int_{C_{\zeta\alpha}} \frac{\psi(z)dz}{(z - \zeta)^{\nu}} \tag{6.4}$$

are still other generalizations of the Hilbert transform.

The methods presented in the previous sections provide inversion formulas for the above transforms. For example let us consider the transform

$$\int_{-\infty}^{\infty} \frac{|\sigma - t|^{\gamma}\psi(t)dt}{\sigma - t} = h(\sigma) \qquad 0 < \gamma < 1 \tag{6.5}$$

which is equivalent to

$$\int_{-\infty}^{\sigma} \frac{\psi(t)dt}{(\sigma - t)^{\nu}} - \int_{\sigma}^{\infty} \frac{\psi(t)dt}{(t - \sigma)^{\nu}} = h(\sigma) \tag{6.6}$$

$$0 < \nu = 1 - \gamma < 1 .$$

This transform was recently the subject of attention of Kober, [12], [13]; Heywood [11]; and Okikiolu [9], [10]. As Kober remarked, the solution of (6.5) by Fourier transform methods is almost trivial if $\psi(t)$ belongs to the Lebesque class $L^1$. This is not the case if $\psi(t)$ is $L^p$, where $p > 1$, due to the divergence of certain integrals; and in order to circumvent this difficulty it is necessary to resort to certain strategic devices. However, we note here that an inversion formula for (6.5) can be found by transforming it into an equation which we have already solved.

The relation

$$t = -\frac{1}{z} - i \quad \text{or} \quad z = -\frac{1}{t+i} \tag{6.7}$$

maps the path in the positive direction along the real axis into the circle C defined by $2iz\bar{z} = z-\bar{z}$ and directed counter clockwise. This circle has its center at $z = i/2$, and it passes through the origin of the z-plane which corresponds to the points at infinity on the t-axis. If we also introduce

$$\sigma = -\frac{1}{\zeta} - i \quad \text{or} \quad \zeta = -\frac{1}{\sigma+i} \tag{6.8}$$

the substitution of (6.7) and (6.8) in (6.6) changes it into

$$\int_{C_{o\zeta}} \frac{\phi(z)dz}{(\zeta-z)^{\nu}} - \int_{C_{\zeta o}} \frac{\phi(z)dz}{(z-\zeta)^{\nu}} = f(\zeta) \tag{6.9}$$

where

$$\phi(z) = z^{\nu-2}\psi(-\frac{1}{z} - i)$$

$$f(\zeta) = \frac{1}{\zeta^{\nu}} h(-\frac{1}{\zeta} - i).$$

From (4.14) the solution of (6.9) is

$$(6.10) \qquad \phi(z) = \frac{1}{2\pi} \tan\frac{\pi\nu}{2} \cdot \frac{d}{dz} \left\{ \begin{array}{l} \displaystyle\int_{C_{oz}} \frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\nu}} \\ \\ \displaystyle + \int_{C_{zo}} \frac{f(\zeta)d\zeta}{(\zeta-z)^{1-\nu}} \end{array} \right\}$$

which by integration, can be expressed in the form

$$(6.11) \qquad \int_{C_{iz}} \frac{\phi(w)dw}{w^{\nu}}$$

$$= \frac{1}{2\pi} \tan\frac{\pi\nu}{2} \cdot \int_{C_{iz}} \frac{1}{w^{\nu}} \frac{d}{dw} \left\{ \begin{array}{l} \displaystyle\int_{C_{ow}} \frac{f(\zeta)d\zeta}{(w-\zeta)^{1-\nu}} \\ \\ \displaystyle + \int_{C_{wo}} \frac{f(\zeta)d\zeta}{(\zeta-w)^{1-\nu}} \end{array} \right\} dw .$$

A simplification of the right hand side of (6.11) yields

$$\int_{C_{iz}} \frac{\phi(w)dw}{w^{\nu}}$$

$$(6.12) \qquad = \frac{1}{2\pi} \tan\frac{\pi\nu}{2} \cdot \left[ \begin{array}{l} \displaystyle\int_{C_{oz}} \frac{f(\zeta)d\zeta}{(\frac{1}{\zeta} - \frac{1}{z})^{1-\nu}\ \zeta^{2-\nu}} + \int_{C_{zo}} \frac{f(\zeta)d\zeta}{(\frac{1}{z} - \frac{1}{\zeta})^{1-\nu}\ \zeta^{2-\nu}} \\ \\ \displaystyle -\int_{C_{oi}} \frac{f(\zeta)d\zeta}{(\frac{1}{\zeta} - \frac{1}{i})^{1-\nu}\ \zeta^{2-\nu}} - \int_{C_{io}} \frac{f(\zeta)d\zeta}{(\frac{1}{i} - \frac{1}{\zeta})^{1-\nu}\ \zeta^{2-\nu}} \end{array} \right].$$

Then, the transformation back to the original variables gives

$$(6.13) \qquad \int_{o}^{t} \psi(\tau)d\tau = \frac{1}{2\pi} \cdot \tan\frac{\pi\nu}{2} \cdot \left\{ \int_{-\infty}^{\infty} \left[\frac{1}{|\sigma-t|^{1-\nu}} - \frac{1}{|\sigma|^{1-\nu}}\right] h(\sigma)d\sigma \right\}$$

for the inversion of (6.6). The formula (6.13) agrees with Kober's [13] result.

## 7. The Kernel $P(\zeta-z)[q(z)\mu(\zeta-z) + r(z)\mu(z-\zeta)]/(\zeta-z)$.

The methods presented in the foregoing sections can be used to solve the equations

$$(7.1) \qquad \int_{L_{\alpha\zeta}} \frac{P(\zeta-z)q(z)\phi(z)dz}{(\zeta-z)^{\nu}} + \int_{L_{\zeta\beta}} \frac{P(\zeta-z)r(z)\phi(z)dz}{(\zeta-z)^{\nu}} = f(\zeta)$$

and

$$(7.2) \qquad \int_{C_{\alpha\zeta}} \frac{P(\zeta-z)q(z)\phi(z)dz}{(\zeta-z)^{\nu}} + \int_{C_{\zeta\alpha}} \frac{P(\zeta-z)r(z)\phi(z)dz}{(\zeta-z)^{\nu}} = f(\zeta)$$

where $P(\zeta-z)$ is the polynomial

$$P(\zeta-z) = \sum_{k=0}^{n} a_k(\zeta-z)^k. \tag{7.3}$$

It is sufficient here to just indicate the procedures which depend on the integral

$$\int_{L_{\alpha\zeta}} \frac{P(\zeta-\lambda)d\lambda}{(\lambda-\alpha)^{1-\nu}(\zeta-\lambda)^{\nu}(\lambda-z)} = \int_{L_{\alpha\zeta}} \frac{\sum_{k=0}^{n} \frac{P^k(\zeta-z)(z-\lambda)^k}{k!}}{(\lambda-\alpha)^{1-\nu}(\zeta-\lambda)^{\nu}(\lambda-z)}\, d\lambda \tag{7.4}$$

$$= P(\zeta-z)\int_{L_{\alpha\zeta}} \frac{d\lambda}{(\lambda-\alpha)^{1-\nu}(\zeta-\lambda)^{\nu}(\lambda-z)} - \int_{L_{\alpha\zeta}} \frac{\sum_{k=1}^{n} \frac{P^k(\zeta-z)(z-\lambda)^{k-1}}{k!}}{(\lambda-\alpha)^{1-\nu}(\zeta-\lambda)^{\nu}}\, d\lambda$$

$$= P(\zeta-z)\int_{L_{\alpha\zeta}} \frac{d\lambda}{(\lambda-\alpha)^{1-\nu}(\zeta-\lambda)^{\nu}(\lambda-z)} - Q(\zeta-z)$$

where

$$Q(\zeta-z) = \int_{L_{\alpha\zeta}} \frac{\sum_{k=1}^{n} \frac{P^k(\zeta-z)[z-\zeta+(\zeta-\lambda)]^{k-1}}{k!}}{(\lambda-\alpha)^{1-\nu}(\zeta-\lambda)^{\nu}}\, d\lambda$$

is a polynomial in $\zeta-z$. The integral (3.2) shows that

$$(7.5)\qquad \int_{L_{\alpha\zeta}} \frac{P(\zeta-\lambda)d\lambda}{(\lambda-\alpha)^{1-\nu}(\zeta-\lambda)^{\nu}(\lambda-z)}$$

$$= \begin{cases} \dfrac{-\pi P(\zeta-z)\cot \pi\nu}{(z-\alpha)^{1-\nu}(\zeta-z)^{\nu}} - Q(\zeta-z) & z \text{ on } L'_{\alpha\zeta} \\ \\ \dfrac{-\pi P(\zeta-z)\, e^{-\pi i\nu}}{(z-\alpha)^{1-\nu}(\zeta-z)^{\nu}\sin \pi\nu} - Q(\zeta-z) & z \text{ not on } L_{\alpha\zeta} \end{cases}.$$

After multiplying (7.5) by $(z-\alpha)^{1-\nu}r(z)\phi(z)$ and integrating with respect to z along $L_{\alpha\beta}$, we find

$$(7.6)\qquad \int_{L_{\zeta\beta}} \frac{P(\zeta-z)r(z)\phi(z)dz}{(\zeta-z)^{\nu}}$$

$$= -\, e^{\pi i\nu}\cos \pi\nu \int_{L_{\alpha\zeta}} \frac{P(\zeta-z)r(z)\phi(z)dz}{(\zeta-z)^{\nu}}$$

$$+ \frac{e^{\pi i\nu}\sin \pi\nu}{\pi} \int_{L_{\alpha\zeta}} \frac{P(\zeta-\lambda)}{(\zeta-\lambda)^{\nu}(\lambda-\alpha)^{1-\nu}} \int_{L_{\alpha\beta}} \frac{(z-\alpha)^{1-\nu}r(z)\phi(z)dzd}{z-\lambda}$$

$$- \frac{e^{\pi i\nu}\sin \pi\nu}{\pi} \int_{L_{\alpha\beta}} (z-\alpha)^{1-\nu}Q(\zeta-z)r(z)\phi(z)dz\ .$$

The elimination of

$$\int_{L_{\zeta\beta}} \frac{P(\zeta-z)r(z)\phi(z)dz}{(\zeta-z)^{\nu}}$$

from (7.6) and (7.1) gives

$$(7.7)\qquad \int_{L_{\alpha\zeta}} \frac{P(\zeta-z)\overset{I}{\phi}(z)dz}{(\zeta-z)^{\nu}} = F(\zeta)$$

where

$$[q(z) - e^{\pi i\nu}r(z)\cos\pi\nu]\phi(z)$$

$$\phi(z) \equiv$$

$$+ \frac{e^{\pi i\nu}\sin\pi\nu}{\pi(z-\alpha)^{1-\nu}} \int_{L_{\alpha\beta}} \frac{(\lambda-\alpha)^{1-\nu}r(\lambda)\phi(\lambda)d\lambda}{\lambda-z}$$

and

$$F(\zeta) \equiv f(\zeta) + \frac{e^{\pi i\nu}\sin\pi\nu}{\pi} \int_{L_{\alpha\beta}} (z-\alpha)^{1-\nu}Q(\zeta-z)r(z)\phi(z)dz.$$

We can now solve (7.7) for $\overset{I}{\phi}(z)$. We have

$$\int_{L_{\alpha z}} \frac{1}{(z-\zeta)^{1-\nu}} \int_{L_{\alpha\zeta}} \frac{P(\zeta-w)\overset{I}{\phi}(w)dw}{(\zeta-w)^{\nu}} = \int_{L_{\alpha z}} \frac{F(\zeta)d\zeta}{(z-\zeta)^{1-\nu}}$$

which reduces to

$$(7.8)\qquad \sum_{k=0}^{n} \frac{a_k\Gamma(\nu)\Gamma(k+1-\nu)}{\Gamma(n+1)} \frac{d^{n-k}}{dz^{n-k}} \int_{L_{\alpha z}} (z-w)^{n}\overset{I}{\phi}(w)dw$$

$$= \int_{L_{\alpha z}} \frac{F(\zeta)d\zeta}{(z-\zeta)^{1-\nu}} .$$

This is an ordinary differential equation for $\int_{L_{\alpha z}} (z-w)^{n}\overset{I}{\phi}(w)dw$.

If we denote the solution of (7.8) by $S(z)$, then

$$\int_{L_{\alpha z}} (z-w)^n \underline{\phi}(w)dw = S(z)$$

and hence

$$[q(z) - e^{\pi i\nu} r(z) \cos \pi\nu]\phi(z)$$

(7.9) $\underline{\phi}(z) \equiv$

$$+ \frac{e^{\pi i\nu} \sin \pi\nu}{\pi(z-\alpha)^{1-\nu}} \int_{L_{\alpha\beta}} \frac{(\lambda-\alpha)^{1-\nu} r(\lambda)\underline{\phi}(\lambda)d\lambda}{\lambda-z}$$

$$= \frac{1}{\Gamma(n+1)} \cdot \frac{d^{n+1}}{dz^{n+1}} \; S(z) \; .$$

Thus if $\phi(z)$ satisfies (7.1) it must also satisfy the Cauchy singular integral equation (7.9). Conversely, if $\phi(z)$ is any solution of (7.9) such that the integrals

$$\int_{L_{\alpha\zeta}} \frac{q(z)P(\zeta-z)\phi(z)dz}{(\zeta-z)^{\nu}} \quad \text{and} \quad \int_{L_{\zeta\beta}} \frac{r(z)P(\zeta-z)\phi(z)dz}{(\zeta-z)^{\nu}}$$

exist then $\phi(z)$ satisfies (7.1).

The equation

$$(7.10) \quad \int_{C_{\alpha\zeta}} \frac{P(\zeta-z)\phi(z)dz}{(\zeta-z)^{\nu}} + k \int_{C_{\zeta\alpha}} \frac{P(\zeta-z)\phi(z)dz}{(\zeta-z)^{\nu}} = f(\zeta)$$

can be solved without solving a Cauchy singular integral equation.

If we follow the procedure of Section 4 we can base the solution of (7.10) entirely on the integral (7.5), another like it, and the solution of generalized Abel equations.

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Some Integral Equations Related to Abel's Equation and the Hilbert Transform

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13. ABSTRACT This report is concerned with the inversion of a generalization of the Hilbert transform

$$\int_a^b \frac{\phi(z)dz}{z-\zeta} = f(\zeta)$$

where $f(\zeta)$ is prescribed. The generalized transform is

$$\int_{L_{\alpha\zeta}} \frac{P(\zeta-z)q(z)\phi(z)dz}{(\zeta-z)^{\nu}} + \int_{L_{\zeta\beta}} \frac{P(\zeta-z)r(z)\phi(z)dz}{(z-\zeta)^{\nu}} = f(\zeta)$$

in which $P(z)$ is a polynomial while $q(z)$, $r(z)$ and the path $L_{\alpha\beta}$ in the complex z-plane are arbitrary.

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